Computers and Mathematics with Applications 58 (2009) 1788–1791

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The general result of Gol'dberg's theorem concerning the growth of meromorphic solutions of algebraic differential equations[☆]

Wenjun Yuan[a,*], Bing Xiao[b], Jianjun Zhang[c,*]

[a] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, 510006, China
[b] Department of Mathematics, Xinjiang Normal University, Wulumuqi, 830054, China
[c] Department of Mathematics, Nanjing University, Nanjing, 210093, China

ARTICLE INFO

Article history:
Received 23 June 2008
Received in revised form 17 December 2008
Accepted 8 July 2009

Keywords:
Meromorphic function
Normal family
Growth order
Spherical derivative
Algebraic differential equations

ABSTRACT

In this paper, we give a general estimate result of Gol'dberg's theorem concerning finite growth order of meromorphic solutions of first-order algebraic differential equations by using the normal family theory. It is an extending result of corresponding theorem for Bergweiler and Barsegian and so on. We also give some examples to show that our result is sharp in special case.

1. Introduction and main result

Let $f(z)$ be a function meromorphic in the complex plane. We use the standard notations of Nevanlinna theory and denotes the order of $f(z)$ by $\lambda(f)$ (see Hayman [1], He [2], Laine [3] and Yang [4]).

Let D be a domain in the complex plane. A family $\mathcal{F}$ of meromorphic functions in D is normal in D, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges locally uniformly by spherical distance to a function $g(z)$ meromorphic in D (function $g(z)$ is permitted to be identically infinity). We define spherical derivative of the meromorphic function $f(z)$ as follows:

$$f^{\sharp}(z) := \frac{|f'(z)|}{1+|f(z)|^2}.$$

An algebraic differential equation for $w(z)$ is of the form

$$P(z, w, w', \dots, w^{(k)}) = 0, \tag{1}$$

where P is a polynomial in each of its variables.

As everyone knows, it is has been one of the important topics to research the growth of meromorphic solution $w(z)$ of differential equation (1) in the complex plane $\mathbb{C}$.

In 1956, Gol'dberg [5] proved the following result.

☆ Foundation item: Supported by the NSF of China (10771220) and Guangzhou Education Bureau (62035).
* Corresponding author.
E-mail addresses: gzywj@tom.com (W. Yuan), zhangjianjun1982@163.com (J. Zhang).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.07.092

Theorem A. *Let $w(z)$ be any meromorphic solution of algebraic differential equation (1) with $k = 1$, then the growth order $\lambda(w)$ of $w(z)$ satisfies $\lambda(w) < \infty$.*

For half a century, Bank and Kaufman [6] (see [3]), Barsegian [7,8], Bergweiler [9] etc. gave some extensions or different proofs, but the results have not changed. In this paper, we extend their results and give a general estimate of order of $w(z)$, which depends on the degrees of coefficients of differential polynomial for $w(z)$. In order to state these results, we first introduce some notations: $n \in \mathrm{N} = \{1, 2, 3, \dots, \}$, $r_j \in \mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$ for $j = 1, 2, \dots, n$, and put $r = (r_1, r_2, \dots, r_n)$. Define $M_r[w]$ by

$$M_r[w](z) := [w'(z)]^{r_1}[w''(z)]^{r_2} \cdots [w^{(n)}]^{r_n},$$

with the convention that $M_{\{0\}}[w] = 1$. We call $p(r) := r_1 + 2r_2 + \cdots + nr_n$ the weight of $M_r[w]$. A differential polynomial $P[w]$ is an expression of the form

$$P[w](z) := \sum_{r \in I} a_r(z, w(z))M_r[w](z), \tag{2}$$

where the a_r are rational in two variables and I is a finite index set. The weight $\deg P[w]$ of $P[w]$ is given by $\deg P[w] := \max_{r \in I} p(r)$. $\deg_{z,\infty} a_r$ denotes the degree at infinity in variable z concerning $a_r(z, w)$. $\deg_{z,\infty} a := \max_{r \in I} \max\{\deg_{z,\infty} a_r, 0\}$.

Theorem B ([7–9]). *Let $w(z)$ be meromorphic in the complex plane, $n \in \mathrm{N}$, $P[w]$ be a differential polynomial with the form (2), $n > \deg P[w]$. If $w(z)$ satisfies the differential equation $[w'(z)]^n = P[w]$, then the growth order $\lambda(w)$ of $w(z)$ satisfies $\lambda(w) < \infty$.*

Our result may be stated as follows.

Theorem. *Let $w(z)$ be meromorphic in the complex plane, $n \in \mathrm{N}$, $P[w]$ be a differential polynomial with the form (2), $n > \deg P[w]$. If $w(z)$ satisfies the differential equation $[w'(z)]^n = P[w]$, then the growth order $\lambda := \lambda(w)$ of $w(z)$ satisfies*

$$\lambda \le 2 + \frac{2 \deg_{z,\infty} a}{n - \deg P[w]}.$$

We notice that $\deg_{z,\infty} a = 0$ when all the $a_r(z, w)(r \in I)$ are constant coefficients rational function in variable w. Therefore, the following is an easy consequence of the above result.

Corollary. *Let $w(z)$ be meromorphic in the complex plane, $n \in \mathrm{N}$, $P[w]$ be a differential polynomial with constant coefficients in variable w and $n > \deg P[w]$. If $w(z)$ satisfies the differential equation $[w'(z)]^n = P[w]$, then $\lambda(w) \le 2$.*

Remark. Yuan–Li–Zhang [10] proved a general result of the above corollary using a different method here.

Example 1 ([2]). Weierstrass elliptic function $w(z)$ satisfies the differential equation $[w'(z)]^2 = 4(w - e_1)(w - e_2)(w - e_3)$,where e_1, e_2 and e_3 are constants. We know that the order of $w(z)$ is $\lambda(w) = 2$. This example shows that the upper bound given by theorem and corollary is sharp.

Example 2. Consider the first-order Briot–Bouquet differential equation

$$P_0(w)(w')^n + \sum_{k=1}^{n} P_k(w)(w')^{n-k} = 0, \tag{3}$$

where $P_k(w)$ is a polynomial in w with constant coefficients. In 1956, Gol'dberg [5] proved that if $w(z)$ is a single-valued solution of Eq. (1), then $P_0(w)$ is a constant and $\lambda(w) \le 2$, it is obvious that this holds for entire as well as meromorphic solutions. This example shows that our estimate is right.

Example 3 ([11]). For $n > 0$, let $w(z) = \cos z^{\frac{n}{2}}$, then $\lambda(w) = \frac{n}{2}$ and w satisfies the following algebraic differential equation:

$$(w')^2 + \frac{n^2}{4} z^{n-2}(w^2 - 1) = 0 \tag{4}$$

when $n = 1$ or 2, $\deg_{z,\infty} a = 0$, and the growth order $\lambda(w)$ of any meromorphic solution $w(z)$ of Eq. (4) satisfies $\lambda(w) \le 2$; when $n \ge 3$, $\deg_{z,\infty} a = n - 2$, and the growth order $\lambda(w)$ of any meromorphic solution $w(z)$ of Eq. (4) satisfies $\lambda(w) \le 2n - 2$.

Example 4. For $n \geq 2$, entire function $w(z) = e^{z^n}$ satisfies the following algebraic differential equation

$$(w')^n = n^{n-1}(n-1)z^{(n-1)^2-1}w^n - n^{n-1}z^{(n-1)^2-n}w^{n-1}w' + (nz^{n-1} + z^{-1})w(w')^{n-1}. \tag{5}$$

We know that the growth order $\lambda(w) = n$, and $\deg_{z,\infty} a = (n-1)^2 - 1$, the growth order $\lambda(w)$ of any meromorphic solution $w(z)$ of Eq. (5) satisfies $\lambda \leq 2(n-1)^2$.

Example 5. For $n \geq 3$, entire function $w(z) = e^{z^n}$ satisfies the following algebraic differential equation

$$(w')^n = n^{n-2}z^{(n-1)^2-n+1}w^{n-1}w'' + (1-n)z^{-1}w(w')^{n-1}. \tag{6}$$

We know that $\deg_{z,\infty} a = (n-1)^2 - n + 1$, and the growth order $\lambda(w)$ of any meromorphic solution $w(z)$ of Eq. (6) satisfies $\lambda \leq 2 + 2[(n-1)^2 - n + 1]$.

2. Main lemmas

In order to prove our result, we need the following lemmas. Lemma 1 is an extending result of Zalcman [12] concerning normal families.

Lemma 1 ([13]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc, α is a real number. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist, for each $-1 < \alpha < 1$,*

(a) *a number $0 < r < 1$;*
(b) *points z_n with $|z_n| < r$;*
(c) *functions $f_n \in \mathcal{F}$;*
(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$, which order is at most 2. In particular, we may choose w_n and ρ_n, such that

$$\rho_n \leq \frac{2}{[f_n^\sharp(w_n)]^{\frac{1}{1+|\alpha|}}}, \quad f_n^\sharp(w_n) \geq f_n^\sharp(0).$$

Lemma 2. *Let $f(z)$ be meromorphic in the complex plane, $\lambda := \lambda(f) > 2$, then for each $0 < \rho < \frac{\lambda-2}{2}$, there exist points $a_n \to \infty (n \to \infty)$, such that*

$$\lim_{n\to\infty} \frac{f^\sharp(a_n)}{|a_n|^\rho} = +\infty. \tag{7}$$

Proof. Suppose that the conclusion of Lemma 2 is not true, then there exists a positive number $M > 0$, such that for arbitrary $z \in \mathrm{C}$, we have

$$f^\sharp(z) \leq M|z|^\rho. \tag{8}$$

By (8) we can get

$$\begin{aligned} A(t,f) &= \frac{1}{\pi}\int\int_{|z|\leq t}[f^\sharp(z)]^2 \mathrm{d}\sigma_z \\ &\leq \frac{M^2}{\pi}\int\int_{|z|\leq t}|z|^{2\rho}\mathrm{d}\sigma_z \\ &= O(|t|^{2\rho+2}). \end{aligned}$$

Thus we obtain an estimate of Ahlfors–Shimizu characteristic function

$$\begin{aligned} T_0(r,f) &= \int_0^r \frac{A(t,f)}{t}\mathrm{d}t \\ &= O(|r|^{2\rho+2}). \end{aligned}$$

Therefore, the order λ of $f(z)$ can be estimated as $\lambda \leq 2\rho + 2$. This is a contradiction with the choice of ρ. $\square$

3. Proof of theorem

Proof. Suppose that the conclusion of theorem is not true, then there exists a meromorphic solution $w(z)$ satisfies the differential equation $[w'(z)]^n = P[w]$, such that

$$\lambda > 2 + \frac{2 \deg\limits_{z,\infty} a}{n - \deg P[w]}. \tag{9}$$

By Lemma 2 we know that for each $0 < \rho < \frac{\lambda-2}{2}$, there exist sequences $a_m \to \infty (m \to \infty)$, such that (7) is right. This implies that the family $\{w_m(z) := w(a_m + z)\}_{m\in\mathrm{N}}$ is not normal at $z = 0$. By Lemma 1, there exist sequences $\{b_m\}$ and $\{\rho_m\}$ such that

$$|a_m - b_m| < 1, \quad \rho_m \to 0, \tag{10}$$

and $g_m(\zeta) := w_m(b_m - a_m + \rho_m\zeta) = w(b_m + \rho_m\zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$, which order is at most 2. In particular, we may choose b_m and ρ_m, such that

$$\rho_m \le \frac{2}{w^\sharp(b_m)}, \quad w^\sharp(b_m) \ge w^\sharp(a_m). \tag{11}$$

According to (7) and (9)–(11), we can get the following conclusion:

For any fixed constant $0 \le \rho < \frac{\lambda-2}{2}$, we have

$$\lim_{m\to\infty} b_m^\rho \rho_m = 0. \tag{12}$$

In the differential equation $[w'(z)]^n = P[w]$ we now replace z by $b_m + \rho_m\zeta$. Assuming that $P[w]$ has the form (2). Then we obtain

$$\rho_m^{-n}[g_m'(\zeta)]^n = \sum_{r\in I} a_r(b_m + \rho_m\zeta, g_m(\zeta))\rho_m^{-p(r)} M_r[g_m](\zeta)$$

namely

$$[g_m'(\zeta)]^n = \sum_{r\in I} \frac{a_r(b_m + \rho_m\zeta, g_m(\zeta))}{b_m^{\deg a_r}} \{b_m^{\frac{\deg a_r}{n-p(r)}} \rho_m\}^{n-p(r)} M_r[g_m](\zeta). \tag{13}$$

Because $0 \le \rho = \frac{\deg a_r}{n-p(r)} \le \frac{\deg a}{n-\deg P[w]} < \frac{\lambda-2}{2}$, $p(r) < n$, by (12) we can get $[g'(\zeta)]^n = 0$ from (13). Thus we get $g(\zeta)$ is a constant. This is a contradiction with the conclusion of what we have obtained. The proof of theorem is complete. □

Acknowledgements

The first author would like to express his sincere thanks to Professor W. Bergweiler for supplying his manuscript. The authors would like to express their sincere thanks to Professor Ming-liang FANG and De-gui YANG for their helpful discussions and suggestions. The authors would like to thank the referee for his/her helpful suggestions and valuable comments to improve the manuscript.

References

[1] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[2] Y.Z. He, X.Z. Xiao, Algebroid Functions and Ordinary Differential Equations. Beijing, 1988.
[3] I. Laine, Nevanlinna Theory and Complex Differential Equations, de Gruyter, Berlin, 1993.
[4] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin, 1993.
[5] A.A. Gol'dberg, On single-valued solutions of first-order differential equations, Ukraïn. Mat. Zh. (8) (1956) 254–261 (in Russian).
[6] S. Bank, R. Kaufman, On meromorphic solutions of first-order differential equations, Comment. Math. Helv. 51 (1976) 289–299.
[7] G. Barsegian, Estimates of derivatives of meromorphic functions on sets of α-points, J. London Math. Soc. 34 (2) (1986) 534–540.
[8] G. Barsegian, On a method of study of algebraic differential equations, Bull. Hong Kong Math. Soc. 2 (1) (1998) 159–164.
[9] W. Bergweiler, On a theorem of Gol'dberg concerning meromorphic solutions of algebraic differential equations, Complex Var. 37 (1998) 93–96.
[10] W.J. Yuan, Z.R. Li, J.J. Zhang, On the estimate of growth of meromorphic solutions of some classes of higher order algebraic differential equations, J. Guangzhou Univ. 8 (2009) 28–31.
[11] L.W. Liao, C.C. Yang, On the growth andfactorization of entire solutions of algebraic differential equations, Ann. Acad. Sci. Fenn. Math. 25 (2000) 73–84.
[12] L. Zalcman, A heuristic principle in complex function theory, Amer. Math. Monthly 82 (1975) 813–817.
[13] L. Zalcman, Normal families: New perspectives, Bull. Amer. Math. Soc. 35 (1998) 215–230.